$$B_{\pm}^{(r)} \equiv (-1)^r U(b-1, \pm\alpha(\omega - U_p k_1)) + (b-\tfrac{1}{2})U(b+1, \pm\alpha(\omega - U_p k_1)) \text{ for } r=0,1 \quad (21a)$$

$$U_p \equiv U(\bar{y}_p) \quad (21b)$$

$$\alpha \equiv (2iM_\infty / U' k_1)^{1/2} \quad (21c)$$

This result shows that the pressure fluctuation at the plate surface $P(\bar{y}_p,k)$ is related to the plate displacement W_p by

$$\frac{P(\bar{y}_p,k)}{\bar{\rho}_p U_\infty^2} = \frac{\omega - U_p k_1}{M_\infty} \frac{A_- B_+^{(1)} - A_+ B_-^{(1)}}{A_- B_+^{(0)} - A_+ B_-^{(0)}} W_p \quad (22)$$

This is the desired result. When it is inserted into Eq. (5a) we obtain through the use of Eq. (5b)

$$\frac{p(x,o,z,t)}{\bar{\rho}_p U_\infty a_\infty} = \left[\frac{\partial}{\partial t} + U_p \frac{\partial}{\partial x}\right] \times \iint_{\text{plate surf.}} K(x-x', z-z')\, w_p(x',z',t)\, dx'\, dz'$$

where

$$K(x,z) \equiv \frac{i}{(2\pi)^2} \iint e^{i(k_1 x + k_3 z)} \times \left[\frac{A_- B_+^{(1)} - A_+ B_-^{(1)}}{A_- B_+^{(0)} - A_+ B_-^{(0)}}\right] dk$$

In applications one often deals with "generalized aerodynamic forces" rather than actual plate pressure. Expressions for these forces can be obtained most directly by inserting Eq. (22) into the formulas given by Dowell.[1]

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Stability of Beam-Reinforced Circular Plates

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CIRCULAR plates loaded in compression are often employed as structural components in engineering systems. Several authors have investigated the elastic stability of circular plates subjected to various loadings and boundary conditions. G. H. Bryan[1] discussed the stability of a planar plate under in-plane compression and presented the solution to the particular problem of a clamped, solid circular plate under a normal, uniform compressive load per unit length of the plate circumference. He employed energy principles and applied a variational method to obtain his solution. Bryan considered symmetric and nonsymmetric buckling modes in his analysis. H. Reismann[2] solved the problem of the buckling of a solid circular plate with an elastic, spring-type, boundary restraint along the plate circumference. Reismann assumed that the buckling load was associated with the radially symmetric mode. Amon and Widera[3] further developed the Reismann case and analyzed the stability of a solid circular plate with a surrounding edge beam of rectangular cross section. These authors also assumed that the plate buckled into the radially symmetric mode.

In the work done by Yamaki,[4] considering the buckling of annular plates under uniform compression on both plate edges, it was found that higher modal shapes often are associated with the critical buckling loads. Phillips and Carney[5] have recently obtained closed form solutions to show that higher buckling modes can be critical in the case of annular plates with edge beams.

In this note, the edge beam and plate configuration shown in Fig. 1 is allowed to buckle into any mode, so that all possible buckled shapes are considered in order to obtain the lowest, or "critical" buckling load.

Theory

The edge beam in Fig. 1 is simply supported along its entire length, as indicated. The support used under the edge beam constrains the beam to remain in its original plane. It is assumed in the analysis that the plate is integrally attached to the edge beam at the outer edge of the plate. The external load that is applied to the structure P_o is a uniformly distributed compressive load per unit of beam length applied to the outer side of the edge beam.

The solid circular plate has a radius a and a constant thickness t. The other material properties which are pertinent to this analysis are A_B the edge beam cross-sectional area E_B the modulus of elasticity of the beam material, I_B the moment of inertia about the horizontal centroidal axis of the beam cross section, I_{20} the moment of inertia about the vertical centroidal axis of the beam cross section, E the modulus of elasticity of the plate material, and σ Poisson's ratio for the plate material.

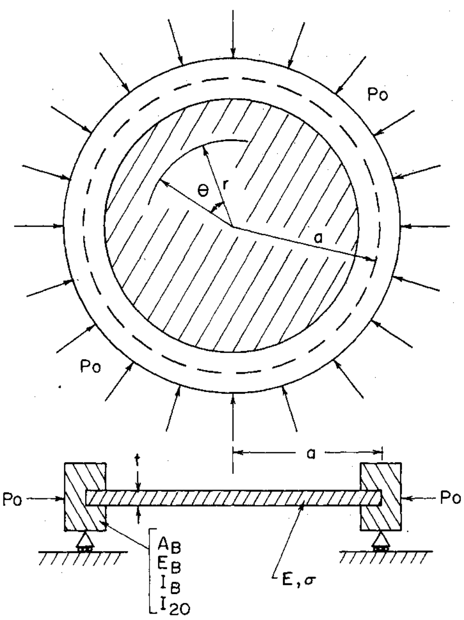

Fig. 1 Plate-edge beam configuration.

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The prebuckling membrane stresses are[5]

$$\sigma_{rr}=2A \quad (1)$$

$$\sigma_{r\theta}=0 \quad (2)$$

$$\sigma_{\theta\theta}=2A \quad (3)$$

where

$$A=-(P_o/2t)\{1/[\alpha(1-\sigma)+1]\} \quad (4)$$

$$\alpha=E_BA'_B/Eat \quad (5)$$

$$A'_B=A_B+(I_{20}/a^2) \quad (6)$$

and E is the modulus of elasticity of the plate.

These stresses are substituted into the field equation for the transverse deflection of the plate,[5] yielding

$$\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}+\left[-\frac{1}{r^2}-\frac{2At}{D}\right]\frac{\partial^2 w}{\partial r^2}+\left[\frac{1}{r^3}-\frac{2At}{Dr}\right]\frac{\partial w}{\partial r}+\left[\frac{4}{r^4}-\frac{2At}{Dr^2}\right]\frac{\partial^2 w}{\partial\theta^2}-\frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial\theta^2}+\frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4}=0 \quad (7)$$

where

$$D=Et^3/12(1-\sigma^2) \quad (8)$$

Finite Fourier trignometric transformations are utilized, along with the Frobenius method, to determine the exact solution to Eq. (7). The result may be expressed in the form[5]

$$w(r,\theta)=C_{o,o}J_o(b_1r)+C_{o,1}Y_o(b_1r)+C_{o,2}+C_{o,3}\ln r+\sum_{n=1}^{\infty}[(C_{n,o}J_n(b_1r)+C_{n,1}Y_n(b_1r)+C_{n,2}r^{-n}+C_{n,3}r^n)\cos n\theta+(C'_{n,o}J_n(b_1r)+C'_{n,1}Y_n(b_1r)+C'_{n,2}r^{-n}+C'_{n,3}r^n)\sin n\theta] \quad (9)$$

where J_n and Y_n are Bessel's functions of the first kind and second kind (Weber's form), each of order n and

$$b_1^2=2At/D \quad (10)$$

The terms that contain the variables $Y_n(b_1r)$, r^{-n}, and ln r in Eq. (9) will give infinite values at $r=0$, hence it becomes necessary to take $C_{o,1}=C_{o,3}=0$, and $C_{n,1}=C'_{n,1}=C_{n,2}=C'_{n,2}=0$ for $n=1,2,3,\ldots,\infty$.

The plate bending boundary conditions which must be satisfied are

$$w(a,\theta)=0 \quad (11)$$

$$\theta^*+\partial w(r,\theta)/\partial r|_{r=a}=0 \quad (12)$$

Equation (12) is due to the radial slope continuity condition between the plate and edge beam at $r=a$, where θ^* is the radial edge beam slope that is found by solving the appropriate ordinary differential equation developed in curved beam theory.[8]

Upon finding θ^*, it is substituted, together with $w(r,\theta)$ as given by Eq. (9), into Eqs. (11) and (12), leading to, for each n a 2×2 determinant that must be set equal to zero so as to yield nontrivial solutions for the P_o loads.

The determinant may be expanded to reveal the following secular equation

$$\left[n^2\frac{G_BI_T}{Da}+\frac{E_BI_B}{Da}\right]J_{n+1}(b_1a)+(2n+1+\sigma)J_{n+1}(b_1a)-(b_1a)J_{n+2}(b_1a)=0 \quad n=0,1,2,3,\ldots \quad (13)$$

where G_B and I_T are the shearing modulus of elasticity and torsion constant, respectively, for the edge beam.

The evaluation of the roots, b_1a, of this secular equation depends upon the mode value n and material properties. The value of n specifies the mode of plate buckling, and for each secular equation there are an infinite number of eigenvalues.

Once a root, b_1a, to Eq. (13) is found it is used to determine the buckling load P_o. It follows from Eqs. (4) and (10) that

$$P_o=(b_1a)^2[\alpha(1-\sigma)+1]D/a^2 \quad (14)$$

The minimum P_o value found for a specific plate, upon admitting all n values, is the critical buckling load.

Results and Conclusions

For all types of edge beam restraint, the critical buckling load for the plate and edge beam has been found to be associated with the $n=0$ buckling mode. Corresponding to the axisymmetric buckling mode, Eq. (13) becomes

$$[\delta+1+\sigma]J_1(b_1a)-(b_1a)J_2(b_1a)=0 \quad (15)$$

where

$$\delta=E_BI_B/Da \quad (16)$$

For the particular case of $\sigma=1/3$, solutions to Eq. (15) are presented in Fig. 2.

Two special cases that are of particular interest in this discussion are a solid circular plate with no edge beam and one with an edge beam that provides the effect of clamping the plate outer edge. The plate with no edge beam represents the case of a simply supported plate on its outer edge. For this case $A_B=I_B=I_{20}=E_B=\alpha=\delta=0$ and the smallest root b_1a to the secular equation is $b_1a=2.07$. Equation (14) then yields $P_o=4.28\,D/a^2$.

The special case of a clamped plate support, where the radial slope of the plate at $r=a$ is identically zero, can be modeled by a special edge beam. As seen in Fig. 1, the P_o ring

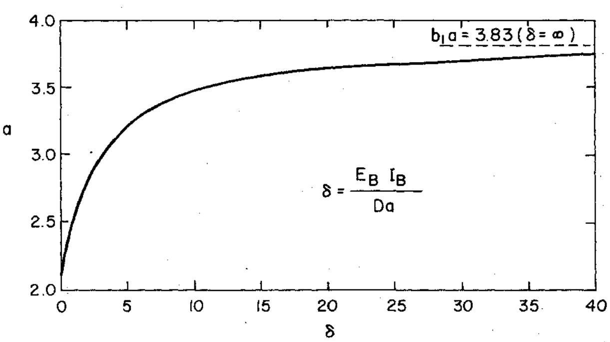

Fig. 2 Dimensionless buckling coefficient (b_1a) vs dimensionless flexural rigidity (δ).

load is applied to the edge beam outer surface. For a clamped edge plate, the P_o load is directly applied to the plate at $r=a$, with no edge beam surrounding the plate. To represent this clamped support case properly, a special type of edge beam is used. The effect described by the E_BA_B' value for the edge beam is a stiffness effect, which provides a direct resistance to the ring load applied to the beam outer edge. Thus, to provide an edge beam that is "equivalent" to the case of a clamped plate, the E_BA_B' value is taken to be zero so that P_o is not resisted by the beam E_BA_B' stiffness. At the same time, a zero slope must be achieved at the plate outer edge. To do this requires the edge beam to possess a very large value of E_BI_B to provide the necessary inertial restraint at the outer edge to keep the radial slope equal to zero. It follows from the definition of α and δ that $\alpha=0$ and $\delta=\infty$ corresponds to an edge beam "equivalent" to the clamped case. It is seen from Fig. 2 that $b_ja=3.83$ for $\delta=\infty$. It follows from Eq. (14) that $P_o=14.68\,D/a^2$ for the clamped case.

The use of an edge beam presents a practical means of increasing the stability of the plate system, leading to savings in weight and materials because: 1) The load P_o is applied directly to the edge beam. The compressive force actually transmitted to the plate is less than P_o. 2) The presence of the edge beam has a stiffening effect on the plate with respect to its rotation capacity at the boundary. These points can best be shown by examples.

Consider a solid circular plate and its surrounding edge beam with the following properties: $a=32$ in. (81.3 cm), $t=1$ in. (2.54 cm), edge beam width $b=1$ in. (2.54 cm), edge beam height $h=4.75$ in. (12.1 cm), $\sigma=1/3$, $E_B=E$. It therefore follows that $\delta=2.98$, $\alpha=0.148$, $b_ja=3.0$ (from Fig. 2), and Eq. (14) gives $P_o=9.88\,D/a^2$. This same plate, if simply supported, buckles at $P_o=4.28\,D/a^2$, and if clamped, has a $P_o=14.68\,D/a^2$ buckling load. If the plate-edge beam system has the values $a=24.5$ in. (62.25 cm), $t=2/3$ in. (1.69 cm), $b=1$ in. (2.54 cm), $h=8.5$ in. (21.60 cm), $E=E_B$ and $\sigma=1/3$, then $\delta=75.2$, $\alpha=0.520$, and $b_ja=3.78$ (from Fig. 2), and Eq. (14) gives $P_o=19.27\,D/a^2$ as the buckling load. For this plate, the force actually transmitted to the plate by the edge beam is $14.29\,D/a^2$. The additional amount of buckling load, $19.27\,D/a^2$ as compared to $14.29\,D/a^2$, is due to the direct stiffness resistance of the edge beam cross section. The force as transmitted to the plate at $r=a$ is still less than the clamped edge case ($14.29\,D/a^2$ as compared to $14.68\,D/a^2$).

The edge beam therefore does provide a practical way to vary the condition of fixity at the outer edge of the plate, and the resulting stability which is achieved in simply attaching an edge beam to the outer edge of the plate can be greater than the stability provided by using the familiar clamped edge support.

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Sustained Small Oscillations in Nonlinear Control Systems

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I. Introduction

CONSIDER the system of first-order equations

$$x' = f(x, \mu) \tag{1}$$

where μ is a parameter, f is assumed analytic in x and μ at the origin of (x, μ) space and $f(0, \mu) = 0$. Suppose there exists a family of solutions

$$\bar{x}(t, \mu) \tag{2}$$

such that to each neighborhood of the origin in (x, μ) there corresponds at least one value, μ, such that $[\bar{x}(t, \mu), \mu]$ is contained in that neighborhood. In this ,case, the origin is called a bifurcation point, the family of solutions $\bar{x}(t, \mu)$ is called a bifurcating branch, and the solutions corresponding to fixed values of the parameter μ are called bifurcating solutions.[3]

Example

Consider the scalar equation $x' = x(x-\mu)$. It can easily be seen that $\bar{x}(t, \mu) = \mu$ is a bifurcating branch of solutions corresponding to a bifurcation branch at the origin of the two-dimensional (x, μ) space and that the bifurcating solutions are constant, or steady-state solutions.

In the following, we shall utilize some results from bifurcation theory to investigate the existence of small amplitude periodic behavior in launch vehicle dynamics. It will be assumed that the nonlinearity exists as a cubic term in the rudder response.

Starting with Poincare, there have been a number of important contributors to the theory. Among the early contributors to the theory of periodic bifurcations were Hopf[2] and Friedrichs.[1] We shall follow quite closely the approach given in Sattinger.[4]

II. Bifurcations in System Theory

In addition to the existence of bifurcating solutions, either steady-state or periodic, it is usually necessary in practice to determine their stability properties. The definition of asymptotic stability of steady-state solutions is well known and need not be presented here. However, the definition of orbital stability of periodic solutions is perhaps less well known and is given here for convenience.

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